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Thermal fluctuations of a quantized massive scalar field in Rindler background

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Abstract

Thermal fluctuations for a massive scalar field in the Rindler wedge are obtained by applying the point-splitting procedure to the zero temperature Feynman propagator in a conical spacetime. Renormalization is implemented by removing the zero temperature contribution. It is shown that for a field of non vanishing mass the thermal fluctuations, when expressed in terms of the local temperature, do not have Minkowski form. As a by product, Minkowski vacuum fluctuations seen by an uniformly accelerated observer are determined and confronted with the literature.

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I. INTRODUCTION

It is well known that certain aspects of the study of quantum fields vibrating near the horizon of a large black hole amount to consider thermal averages in the Rindler wedge [1,2]. Such a connection is one of the reasons which makes the latter a topic of much interest [3,4].

In this letter, the thermal fluctuations $\langle\phi^2(x)\rangle$ for a massive scalar field in the Rindler wedge are worked out by applying an algorithm which consists of reading Rindler thermal averages from vacuum averages in a conical spacetime [5–7]. In Sec. II, the thermal fluctuations in Minkowski spacetime are presented in order to be compared with those in the Rindler wedge. Sec. III explains the algorithm mentioned above. In Sec. IV, $\langle\phi^2(x)\rangle$ in four dimensional Rindler background are determined in various temperature regimes, and for small and large masses of the field. As quantities in Sec. IV are renormalized by subtracting their zero temperature values (the Rindler vacuum contribution), for a particular temperature (the Hawking temperature) they should reproduce Minkowski vacuum fluctuations as seen from an uniformly accelerated frame. This fact is verified in Sec. V. Final remarks are presented in Sec VI, and an appendix, where the Euler-Maclaurin formula is used to perform a summation, closes the letter.

Throughout the text, units are such that $\hbar = c = k_B = 1$.

II. THERMAL $<\phi^2>$ IN MINKOWSKI SPACETIME

In Minkowski spacetime, the Feynman propagator for a scalar field $\phi(x)$, of mass M, in thermal equilibrium with a heat bath at temperature T is given by (see e.g. Ref. [8])

$$G_{\mathcal{F}}(x,x') = G^0_{\mathcal{F}}(x,x') - i\int_{-\infty}^{\infty} \frac{d^4\mathrm{k}}{(2\pi)^3}\ \exp\{-i\mathrm{k}(\mathrm{x}-\mathrm{x}')\}\ \frac{\delta(\mathrm{k}^2-M^2)}{e^{|\mathrm{k}_0|/T}-1}, \tag{1}$$

where $G^0_{\mathcal{F}}(x,x')$ is the familiar zero temperature Feynman propagator. According to the point-splitting procedure [1], the average $<\phi^2>$ can be obtained from the Feynman propagator $G_{\mathcal{F}}(x,x')$ by considering

$$\langle\phi^2(x)\rangle = i\lim_{x'\to x} G_{\mathcal{F}}(x,x'), \tag{2}$$

where renormalization is implemented by dropping the zero temperature contribution. Thus one considers Eq. (1) in Eq. (2), and after some manipulations it results

$$\langle\phi^2(x)\rangle = \frac{1}{2\pi^2}\int_0^{\infty} \frac{(\omega^2+2M\omega)^{1/2}}{e^{M/T}e^{\omega/T}-1}\, d\omega. \tag{3}$$

For high temperatures ($M/T \ll 1$) and low temperatures ($M/T \gg 1$) , $\langle\phi^2(x)\rangle$ behaves as (γ is the Euler constant)

$$\frac{T^2}{12} - \frac{MT}{4\pi} - \frac{M^2}{8\pi^2}\left(\log\frac{M}{4\pi T} + \gamma - \frac{1}{2}\right) \tag{4}$$

and

$$M^{1/2}\left(\frac{T}{2\pi}\right)^{3/2} \exp\left\{-\frac{M}{T}\right\}, \tag{5}$$

respectively.

III. THERMAL AVERAGES IN RINDLER BACKGROUND

In order to obtain thermal effects at temperature $1/2\pi\alpha$ in the quantum mechanics of fields in Rindler background (the right Rindler wedge, for which ρ below is positive),

$$ds^2 = \rho^2 d\eta^2 - d\rho^2 - dx^2 - dz^2, \tag{6}$$

one analytically continues the time coordinate η to imaginary values, taking $\varphi := i\eta$ periodic with period $2\pi\alpha$. Considering further $t := ix$, Eq. (6) leads to the geometry of a four dimensional conical spacetime. Namely, the line element is recast as $ds^2 = dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2$ and the identification $(t, \rho, \varphi, z) \sim (t, \rho, \varphi + 2\pi\alpha, z)$ holds. Hence it seems that one can read from vacuum averages in conical spacetime thermal averages in the Rindler wedge. Indeed this is the case, as has been shown in Refs. [5–7]. It happens that the zero temperature propagator $G_{\mathcal{F}}^{(\alpha)}(x, x')$ computed in a conical spacetime may be interpreted as Rindler thermal propagator for temperature $1/2\pi\alpha$. It follows that since $G_{\mathcal{F}}^{(\alpha=1)}(x, x')$ is the usual propagator in Minkowski spacetime, this propagator is a Rindler thermal propagator for temperature $1/2\pi$. Therefore the Minkowski vacuum, which is a pure state, is also a Rindler thermal state (more precisely a statistical mixture), a fact fairly well known.

IV. THERMAL $< \phi^2 >$ IN RINDLER BACKGROUND

Usually quantities in a conical spacetime are renormalized with respect to the Minkowski vacuum ($\alpha = 1$ contribution). In the present context, it seems more natural to renormalize them with respect to the Rindler vacuum (zero temperature contribution, $\alpha \to \infty$) instead [6,7] (perhaps the best motivation for doing so resides in the fact that Rindler thermal potentials coincide with those obtained by a counting of states procedure [4]). One implements that simply by subtracting from $G_{\mathcal{F}}^{(\alpha)}(x, x')$ its value for $\alpha \to \infty$, before using prescription (2), resulting

$$\langle \phi^2(x) \rangle = i \lim_{x' \to x} \left[G_{\mathcal{F}}^{(\alpha)}(x, x') - G_{\mathcal{F}}^{(\alpha=\infty)}(x, x') \right]. \tag{7}$$

Small masses

It has been shown in Ref. [9] that the Feynman propagator for a massive scalar field in an N-dimensional conical spacetime is given by

$$\begin{aligned} G_{\mathcal{F}}^{(\alpha)}(\Delta) = {} & \frac{1}{i\alpha (2\pi^{1/2})^N \rho^{N-2}} \\ & \times \sum_{m=-\infty}^{\infty} e^{im\Delta/\alpha} \left\{ \frac{\Gamma\left[(N-2)/2 + |m|/\alpha\right] \Gamma\left[(3-N)/2\right]}{\pi^{1/2} \Gamma\left[(4-N)/2 + |m|/\alpha\right]} \right. \\ & \qquad \times {}_1F_2 \left[\frac{3-N}{2}; \frac{4-N}{2} - \frac{|m|}{\alpha}, \frac{4-N}{2} + \frac{|m|}{\alpha}; (M\rho)^2 \right] \\ & \qquad + 2^{-2|m|/\alpha} (M\rho)^{N-2(1-|m|/\alpha)} \frac{\Gamma\left[(2-N)/2 - |m|/\alpha\right]}{\Gamma\left[1 + |m|/\alpha\right]} \\ & \qquad \left. \times {}_1F_2 \left[\frac{1}{2} + \frac{|m|}{\alpha}; \frac{|m|}{\alpha} + \frac{N}{2}, 1 + \frac{2|m|}{\alpha}; (M\rho)^2 \right] \right\}, \end{aligned} \tag{8}$$

where ${}_1F_2\left[a;b,c;z\right]$ denotes the generalized hypergeometric function, $\Delta := \varphi - \varphi'$ and one has set $t = t'$, $\rho = \rho'$ and $z = z'$. In a four-dimensional background ($N = 4$), when α is not very large and $M\rho \ll 1$, in order to obtain mass corrections of order $(M\rho)^2$, one has to consider in the second term inside the curly brackets of Eq. (8) only the contribution corresponding to $m = 0$. Then dimensional regularization leads to [9] (terms which vanish in the limit $\Delta \to 0$ will be omitted)

$$G^{(\alpha)}_{\mathcal{F}}(\Delta) = \text{u.v.} - \frac{i}{48\,\pi^2\alpha^2\rho^2} - \frac{iM^2}{8\pi^2}\left[\frac{1}{\alpha}\left(\log\frac{M\rho}{2} + \gamma - \frac{1}{2}\right) - \log\alpha\right], \tag{9}$$

where

$$\text{u.v.} := -\frac{i}{4\pi^2\rho^2\Delta^2} - \frac{iM^2}{8\pi^2}\log\Delta \tag{10}$$

gives rise to the ultraviolet divergences when the limit in Eq. (7) ($\Delta \to 0$) is taken into account. Using Eq. (9) in Eq. (7) when $M = 0$ one reproduces the known expression, $<\phi^2>= 1/48\,\pi^2\alpha^2\rho^2$. Taking into account that the local (Tolman) temperature $\mathcal{T} := 1/\beta\sqrt{g_{00}}$ (where $1/\beta$ is the heat bath temperature [10]) in the Rindler wedge is given by

$$\mathcal{T} := \frac{1}{2\pi\alpha\;\rho}, \tag{11}$$

the massless $<\phi^2>$ can be recast as $\mathcal{T}^2/12$, and hence is Minkowski (cf. Eq. (4)). For the massive case Eq. (7) seems to lead to an inconsistency due to the presence of the term containing $\log\alpha$ in Eq. (9) which diverges in the limit $\alpha \to \infty$. The following analysis shows that when $\alpha \to \infty$ new terms of order $(M\rho)^2$ arise in Eq. (8), compensating the logarithmic divergence and yielding a finite result.

In the limit of very large α not only the $m = 0$ contribution that arises from the second term inside the curly brackets of Eq. (8) is relevant to compute mass corrections of order $(M\rho)^2$. One must also consider (for $N = 4$)

$$\frac{1}{i\alpha(4\pi)^2\rho^2} \times E(\Delta), \tag{12}$$

where

$$\begin{aligned} E(\Delta) &:= \sum_{m\neq 0} e^{im\Delta/\alpha} 2^{-2|m|/\alpha} (M\rho)^{2(1+|m|/\alpha)} \frac{\Gamma\left[-1-|m|/\alpha\right]}{\Gamma\left[1+|m|/\alpha\right]} \\ &= 2(M\rho)^2 \sum_{m=1}^{\infty} a_m \cos\frac{m\Delta}{\alpha}, \end{aligned} \tag{13}$$

with

$$a_m := \frac{\pi\,\mathrm{csc}\{m\pi/\alpha\}}{(1+m/\alpha)\Gamma^2[1+m/\alpha]}\left(\frac{M\rho}{2}\right)^{2m/\alpha}. \tag{14}$$

In order to determine $E(\Delta)$ for large values of α, one applies the method outlined in the Appendix. Then adding Eq. (12) to Eq. (9) it follows (only terms up to $1/\alpha^2$ order will be kept)

$$
G_{\mathcal{F}}^{(\alpha)}(\Delta) = \text{u.v.}
$$

$$
-\frac{i}{48\,\pi^2\alpha^2\rho^2} + \frac{iM^2}{8\pi^2}\left\{\frac{1}{6\alpha^2}\left[\left(\log\frac{M\rho}{2}+\gamma-\frac{1}{2}\right)^2+\frac{1}{4}\right]+\log\left(-\log\left(\frac{M\rho}{2}\right)^2\right)\right\}, \tag{15}
$$

which is clearly finite for $\alpha \to \infty$, yielding

$$
G_{\mathcal{F}}^{(\alpha=\infty)}(\Delta) = \text{u.v.} + \frac{iM^2}{8\pi^2}\log\left(-\log\left(\frac{M\rho}{2}\right)^2\right). \tag{16}
$$

Therefore for a massive field one has two regimes of heat bath temperature $1/2\pi\alpha$ to investigate. When α is not very large, by subtracting Eq. (16) from Eq. (9), and then applying prescription (7), it results

$$
\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{12} + \frac{M^2}{8\pi^2}\left[2\pi\left(\log\frac{M\rho}{2}+\gamma-\frac{1}{2}\right)\rho\mathcal{T} + \log\left(-4\pi\rho\mathcal{T}\log\frac{M\rho}{2}\right)\right], \tag{17}
$$

where Eq. (11) has been observed. When α is very large, by subtracting Eq. (16) from Eq. (15), Eq. (7) leads to

$$
\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{12}\left\{1-(M\rho)^2\left[\left(\log\frac{M\rho}{2}+\gamma-\frac{1}{2}\right)^2+\frac{1}{4}\right]\right\}. \tag{18}
$$

Noticing the equality

$$
\frac{M}{\mathcal{T}} = M\rho\;2\pi\alpha, \tag{19}
$$

it follows that Eq. (17) should be compared with Eq. (4), and Eq. (18) should be compared with Eq. (4) or Eq. (5) (depending on how large α is). It is clear that even for small masses the behaviour of the Rindler thermal fluctuations departs from the Minkowski type.

Large masses

For large masses ($M\rho \gg 1$) Eq. (8) is not very handy to evaluate thermal fluctuations. It is rather more convenient to work with expression [11]

$$
\begin{aligned}
G_{\mathcal{F}}^{(\alpha)}(\Delta) = G_{\mathcal{F}}^{(\alpha=1)}(\Delta) &+ \frac{iM}{16\,\pi^3\alpha\rho}\int_0^\infty d\omega\;\frac{K_1\left(2M\rho\cosh\{\omega/2\}\right)}{\cosh\{\omega/2\}} \\
&\times\left[\frac{\sin\{(\Delta+\pi)/\alpha\}}{\cosh\{\omega/\alpha\}-\cos\{(\Delta+\pi)/\alpha\}} - \frac{\sin\{(\Delta-\pi)/\alpha\}}{\cosh\{\omega/\alpha\}-\cos\{(\Delta-\pi)/\alpha\}}\right],
\end{aligned} \tag{20}
$$

with K_1 denoting the modified Bessel function of the second kind, and $\alpha > 1/2$ (note that Eq. (20) cannot be used to investigate the behaviour of $\langle\phi^2(x)\rangle$ for heat bath temperatures $1/2\pi\alpha \geq 1/\pi$). Now, noting that for very large α (in leading order)

$$
\frac{\sin\{\pi/\alpha\}}{\alpha\left(\cosh\{\omega/\alpha\}-\cos\{\pi/\alpha\}\right)} = \frac{2\pi}{\pi^2+\omega^2} - \frac{\pi}{6\alpha^2}, \tag{21}
$$

Eq. (7) leads to

$$\langle\phi^2(x)\rangle = \frac{M}{8\pi^3\rho}\int_0^\infty d\omega\;\frac{K_1\left(2M\rho\cosh\{\omega/2\}\right)}{\cosh\{\omega/2\}}\left[\frac{\sin\{\pi/\alpha\}}{\alpha\left(\cos\{\pi/\alpha\}-\cosh\{\omega/\alpha\}\right)}+\frac{2\pi}{\pi^2+\omega^2}\right]. \quad (22)$$

A closed expression for the Rindler thermal fluctuations at low heat bath temperatures $1/2\pi\alpha$ can be obtained by inserting Eq. (21) in Eq. (22). In so doing, the integration over ω can be performed [12] yielding

$$\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{12}(M\rho)^2\left[K_1^2(M\rho)-K_0^2(M\rho)\right], \quad (23)$$

where Eq. (11) has been observed. (As a check of consistency, by considering the behaviour of $K_\nu(z)$ for small z in Eq. (23), Eq. (18) is recovered.). When $M\rho \gg 1$, one reads from Eq. (23) a non Minkowski thermal fluctuation (cf. Eq (5))

$$\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{24}\pi\exp\{-2M\rho\}. \quad (24)$$

The method of steepest descent [13] can be used in the limit of large $M\rho$ to perform the integration in Eq. (22). Accordingly, considering the large z behaviour of $K_\nu(z)$ and Eq. (11), one is led to

$$\langle\phi^2(x)\rangle = \frac{1}{8\pi^3\rho^2}\left(1-\pi^2\rho\mathcal{T}\cot\{\pi^2\rho\mathcal{T}\}\right)\exp\{-2M\rho\}, \quad (25)$$

which also departs from the Minkowski type. (By making α very large in Eq. (25), Eq. (24) is consistently reproduced.).

V. ACCELERATED FRAMES

The world line $x(\tau)$ (τ is the proper time) of an uniformly accelerated frame, with proper acceleration a, can be expressed in terms of Rindler coordinates (see Eq. (6)), where

$$\eta(\tau) = a\tau \qquad\qquad \rho(\tau) = 1/a \quad (26)$$

(x and z are kept constant). The Minkowski vacuum fluctuations $\langle 0_M|\phi^2(\tau)|0_M\rangle$ as seen from this frame can formally be obtained by considering the Minkowski two point function $g(\tau-\tau') := \langle 0_M|\phi(\tau)\phi(\tau')|0_M\rangle$ evaluated along the world line, and in the limit $\tau'\to\tau$ (i.e., $x'\to x$). Such a procedure requires renormalization, which is taken as being

$$\langle\phi^2(x)\rangle = \lim_{x'\to x}\left[\langle 0_M|\phi(x)\phi(x')|0_M\rangle - \langle 0_R|\phi(x)\phi(x')|0_R\rangle\right], \quad (27)$$

i.e., one subtracts the Rindler vacuum contribution. If $F(\omega)$ is the Fourier transform of $g(\tau-\tau')$, it can be shown that [14]

$$\langle\phi^2(x)\rangle = \frac{1}{\pi}\int_0^\infty d\omega\;F(\omega). \quad (28)$$

(The Fourier transform $F(\omega)$ has been evaluated in Chapter 4 of Ref. [14].)

As careful manipulations (which will be omitted here since they are lengthy) show, by inserting in Eq. (28) the expression for $F(\omega)$ in the limit of small masses (incidentally, it should be remarked that this limit is mistakenly evaluated in Ref. [14]), one ends up with

$$\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{12} + \frac{M^2}{8\pi^2}\left[\log\frac{M}{4\pi\mathcal{T}} + \gamma - \frac{1}{2} + \log\left(-\log\left(\frac{M}{4\pi\mathcal{T}}\right)^2\right)\right], \tag{29}$$

where

$$\mathcal{T} := \frac{a}{2\pi}. \tag{30}$$

Recalling Eq. (26), it follows that Eq. (29) is just Eq. (17) for $\alpha = 1$. In the limit of large masses [14], Eq. (28) leads to

$$\langle\phi^2(x)\rangle = \frac{\mathcal{T}^2}{2\pi}\exp\left\{-\frac{M}{\pi\mathcal{T}}\right\} \tag{31}$$

instead. One sees that Eq. (31) follows from Eq. (25) when $\alpha = 1$.

Such identifications should not be seen as unexpected. Indeed, when $\alpha = 1$, Eq. (7) is identical to Eq. (27).

VI. FINAL REMARKS

By comparing Eq. (29) with Eq. (4) and Eq. (31) with Eq. (5), one concludes that the statement "an accelerated observer sees the Minkowski vacuum as a (Minkowski) thermal bath whose temperature is proportional to the proper acceleration" is not correct if the scalar field has a non vanishing mass (it should be mentioned that this result has also been reached in Ref. [14] by comparing the Fourier transforms of the corresponding two-point functions). In fact this statement does not hold in a variety of situations (see Refs. [14–16] and references therein).

The fact that the zero temperature propagator in a conical spacetime can be interpreted as a Rindler thermal propagator does not merely follow as a consequence of recasting Eq. (6) as the line element in a conical spacetime. A crucial aspect is the requirement of finiteness of the eigenmodes at the conical singularity, i.e. on the Rindler horizon $\rho = 0$. In fact, all the conical eigenmodes vanish at the conical singularity, except the zero angular momentum eigenmode (see Ref. [9] and references therein). Therefore when the field $\phi(x)$ is expanded in terms of these conical eigenmodes, the net effect is essentially the one corresponding to the Dirichlet boundary condition. Rindler thermal Green functions [5,6,17], on the other hand, are built up with Rindler eigenmodes which, unlike the conical eigenmodes, oscillate very rapidly as the horizon is approached. Nevertheless, the net effect of these oscillations in the field is again the one corresponding to Dirichlet's boundary condition [18]. (At this point it should be mentioned that Eq. (27) has been evaluated in Ref. [18] by using the Schrödinger formalism. When $\alpha = 1$, the first term inside the brackets in Eq. (22) drops, and the expression for $\langle\phi^2(x)\rangle$ in Ref. [18] is reproduced.)

¿From the first paragraph of Sec. IV, one clearly sees that the Rindler thermal fluctuations can be obtained from the corresponding conical vacuum fluctuations (which are renormalized with respect to the Minkowski vacuum) by subtracting from the latter their values for $\alpha = \infty$. When such a recipe is applied to the expression for $\langle\phi^2(x)\rangle$ in Refs. [9,19], a logarithmic divergence arises. As shown in Sec. IV above, this is just an apparent inconsistency, since the divergence cancels out when the limit $\alpha \to \infty$ is properly taken. By applying this recipe to the expression for the large mass behaviour of the vacuum fluctuations in Ref. [20], Eq. (25) is reproduced up to a factor 1/2 (which is missing in Ref. [20]).

A last remark concerning Eq. (25) (and also its conical counterparts in Refs. [20,21]) is in order. When $\mathcal{T}$ approaches $1/\pi\rho$ from below, the right hand side of Eq. (25) grows without bounds. The reason for such a behaviour may be just a consequence of the fact that the limit of validity of Eq. (20) is approached ($\alpha = 1/2$), or it may be related with some other as yet obscured mathematical reason (e.g., some limitation in the method of steepest descent). This fact requires further analysis.

An investigation along these lines concerning the energy momentum tensor will appear elsewhere.

The approach in this letter may be useful in the investigation of quantum processes taking place in massive Rindler thermal baths (e.g. Ref. [22]).

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APPENDIX

One looks for an asymptotic expansion in powers of $1/\alpha$ for the series

$$\frac{E(\Delta)}{2(M\rho)^2\alpha} = \sum_{k=1}^{\infty} H_\alpha(k) \tag{32}$$

where

$$H_\alpha(x) := \frac{\pi}{\alpha\sin\{x\pi/\alpha\}} \frac{(M\rho/2)^{2x/\alpha}\cos\{x\Delta/\alpha\}}{\Gamma[1+x/\alpha]\Gamma[2+x/\alpha]} \tag{33}$$

The Euler-Maclaurin formula (see, for example, Chapter 7 in Ref. [23]) is appropriate for this purpose and makes no use of any regularization technique. It may be written as

$$\sum_{k=a}^{a+r} G(k) = \int_a^{a+r} G(x)\; dx + \frac{1}{2}(G(a+r)+G(a)) + \frac{1}{12}(G'(a+r)-G'(a)) + \\ + \frac{1}{6}\int_0^1 \phi_3(x) \sum_{k=a}^{a+r-1} G'''(k+x)\; dx,$$

where $G(x)$ is an analytic function in the interval $[a, a+r]$, a and r are integers (r being positive) and $\phi_3(t)$ is the third degree Bernoulli polynomial. When applied to the function $H_\alpha(x)$ defined in Eq. (33) for the interval $(0,\infty)$, and afterwards expanding in powers of $1/\alpha$, the Euler-Maclaurin formula leads to

$$\sum_{k=1}^{\infty}{}^{\#} H_\alpha(k) \sim \log\alpha + C(M\rho/2,\Delta) - \frac{1}{\alpha}\left[\log\frac{M\rho}{2} + \gamma - \frac{1}{2} + \mathcal{O}(M\rho)\right]$$
$$-\frac{1}{6\alpha^2}\left[\left(\log\frac{M\rho}{2} + \gamma - \frac{1}{2}\right)^2 + \frac{1}{4} - \frac{\Delta^2}{4} + \mathcal{O}(M\rho)\right]$$
$$+\mathcal{O}(1/\alpha^3)\cdot\mathcal{O}(\log^3(M\rho/2)), \tag{34}$$

where

$$C(M\rho/2,\Delta) = \gamma - PV\int_{0^+}^{\infty} \log u\,\frac{d}{du}\left[\frac{\pi u}{\sin\{\pi u\}}\frac{(M\rho/2)^{2u}\cos\{\Delta u\}}{\Gamma[1+u]\Gamma[2+u]}\right]du$$
$$-2\left(\frac{7}{12} - \gamma\right)\sum_{k=1}^{\infty}(-1)^k\frac{(M\rho/2)^{2k}\cos\{k\Delta\}}{\Gamma[1+k]\Gamma[2+k]}. \tag{35}$$

Some remarks concerning Eqs. (34) and (35) are in order. The symbol $\sim$ indicates that the expansion is to be understood only in the assymptotic sense (see e.g. Ref. [13]). The # in the summation specifies that when α is an integer, the summation is done over every positive integer k, except multiples of α. The "PV" in front of the integration denotes its Cauchy Principal Value. These subtleties arise due to the fact that the function $H_\alpha(x)$ defined in Eq. (33) diverges for any x which is a multiple of α . The Euler-Maclaurin formula used to derive Eq. (34) has been carefully manipulated to take into account these discontinuities.

It is shown below that, for $M\rho \ll 1$, $C(M\rho/2,0)$ behaves as

$$C(M\rho/2,0) = -\log(-\log(M\rho/2)^2). \tag{36}$$

Writing Eq. (35) for $\Delta = 0$ in terms of

$$g(u) := \frac{\pi u}{\sin\{\pi u\}}\frac{1}{\Gamma[1+u]\Gamma[2+u]} \quad \text{and} \quad (M\rho/2)^{2u}, \tag{37}$$

integrating by parts and rearranging the terms, one has that

$$C(M\rho/2,0) = \gamma - g(0^+)\int_{0^+}^{1}\frac{1-(M\rho/2)^{2u}}{u}du \; + g(0^+)\int_{1}^{\infty}\frac{(M\rho/2)^{2u}}{u}du$$
$$+\int_{0^+}^{\infty}\frac{g(u)-g(0^+)}{u}(M\rho/2)^{2u}du \; + \mathcal{O}((M\rho/2)^2), \tag{38}$$

where the $\mathcal{O}((M\rho/2)^2)$ is the leading behaviour of the summation in Eq. (35) for small values of $M\rho$.

The integrations in Eq. (38) behave, for small values of $M\rho$, as

$$\int_{0^+}^{1}\frac{1-(M\rho/2)^{2u}}{u}du = \log(-\log(M\rho/2)^2) + \gamma + \mathcal{O}(-\frac{(M\rho)^2}{\log M\rho}),$$
$$\int_{1}^{\infty}\frac{(M\rho/2)^{2u}}{u}du = \mathcal{O}(-\frac{(M\rho)^2}{\log M\rho}),$$
$$\int_{0^+}^{\infty}\frac{g(u)-g(0^+)}{u}(M\rho/2)^{2u}du = \mathcal{O}(-\frac{1}{\log M\rho}). \tag{39}$$

Thus, replacing $g(0^+) = 1$ and Eqs. (39) in Eq. (38), one finally ends up with Eq. (36).

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